Kentucky Journal of Mathematics Teacher Education
2023, Volume 2: Issue 1, pp. 7-19

The Case for Thinking Deeply About Simple Things

Catherine Pullin Lane
Baldwin Wallace University

Abstract

What topics do our preservice teachers understand at a conceptual level? How can we provide opportunities for our students to take deep dives into basic, or 'simple' concepts? At our university we created a class for preservice teachers to explore Calculus concepts at a more leisurely pace. This paper retells what happened when a group of preservice teachers were given an initial exploration into the difference quotient. A planned one-day discussion grew into a weeklong conversation that provided them the opportunity to expand their experience with, and knowledge of, limits.

Keywords: conceptual understanding, limits, calculus

Addressing the Need

The Association of Mathematics Teacher Educators (AMTE) states that many preservice teachers "will have experienced success with a narrow school mathematics curriculum that did not promote conceptual knowledge or emphasize mathematical practices and process" (2017, p. 120). It appears that many of the math courses at our university have the same weaknesses. We are an independent, liberal arts and sciences university located 20 miles outside a large metropolitan city serving 5,500 combined graduate and undergraduate students. Even though our preservice teachers had completed a three-course sequence in Calculus and then a Real Analysis course, it became clear through conversations that they did not have a clear grasp of the underlying concepts. When it came to Calculus, our students were more than proficient at calculating derivatives and integrals, but their conceptual knowledge was lacking. Preservice teachers had difficulty discussing Calculus beyond the algorithmic procedures. For example, students could not say more about the first derivative than it told them if the graph was increasing or decreasing. Few of the mathematics courses the preservice teachers had taken had provided them with the opportunity to explore the underlying concepts or to pose their own questions for investigation. Our solution was to create a course specifically designed to give prospective teachers the space to do this.

Calculus Concepts for Teachers

Professional organizations recommend that teachers develop conceptual understanding, or a deep understanding, of the mathematics that they teach, though a definition of those phrases is difficult to find (AMTE, 2017; Conference Board of the Mathematical Sciences [CBMS], 2012; National Council of Teachers of Mathematics [NCTM], 2012). Former NCTM president Trena Wilkerson captures what is generally meant by these phrases and describes a deep understanding of mathematics as understanding that "goes beyond algorithms, procedures, and knowledge" (2022). The *Mathematical Education of Teachers II (MET II)* recommends that prospective teachers enroll in courses that allow for a deeper look at high school mathematics concepts (CBMS, 2012). Therefore, one can interpret this as a recommendation for a course in which students would look at topics with a focus beyond simply getting the answers to a set of problems. This recommendation helped build the case for our new course, Calculus Concepts for Teachers (CCT). This course was developed in the spirit of the elective courses encouraged by *MET II* along with Arnold Ross's motto that we should "Think deeply of simple things" (Jackson, 2001), that is, think about topics in Calculus beyond learning the processes for calculating limits, derivatives, and integrals.

As the faculty member of the Mathematics & Statistics department hired to primarily teach mathematics content courses for our preservice teachers, I worked with my colleague in the Education Department as the course took form. While I would be the person teaching the course, its creation was a joint effort. The primary goal of CCT was to provide students with the freedom to explore the concepts they encountered in Precalculus and Calculus. The plan was to incorporate hands-on activities and explorations that would illuminate the Calculus concepts. To keep the focus on the concepts rather than the computation, calculations such as derivatives and integrals, would be handled by technology. A potential bonus from this decision would be the added time students would have working with technology, either Desmos or a graphing calculator. We wanted students to feel confident that they could clearly communicate the Calculus concepts to a typical high school student at the completion of CCT.

Calculus Concepts for Teachers was offered for the first time in the fall of 2020. The course had been developed before the COVID-19 pandemic. By the time the course was set to begin, infections were on the rise and vaccines were not yet available. Our university pandemic protocols meant that at each class meeting, half of the students were in the physical classroom while the other half were online joining the classroom via Zoom. Additionally, a few students had university permission to be online for the entire semester. For the first few weeks of the semester, I attempted to proceed with the originally planned hands-on activities. For example, one day we set up Hot Wheels tracks in a hallway to investigate and model mathematically how the height of the ramp impacted the distance the car travelled. All students were logged into Zoom and by placing multiple laptops in the hallway we tried to help the online students share in the experience. This arrangement with the online students watching the hands-on activities, did not provide students the opportunity to engage in the classroom as the online students rarely joined in the discussions at the end of each class. Therefore, I looked for concepts that we as a class could explore using technology instead. This was not part of the original course design, and I did not have time to develop the activities in depth before they were used in class. The following is a retelling of how what was intended to be a one-day discussion of the difference quotient, $\frac{f(x+h)-f(x)}{h}$, spurred a weeklong exploration in which students wrestled with their understanding of limits.

Thinking Deeply About Limits

According to Ohio state standards (Ohio Department of Education, 2017), students are generally introduced to functions and the calculation of the slope of a line in the eighth grade. In some later course, perhaps Algebra II or Precalculus, students are introduced to the difference quotient, which is the slope of a secant line, by using function notation. If introduced in a College Algebra course, students are asked to simplify the difference quotient for a function and are perhaps told that this will be used in Calculus. In Calculus, the difference quotient is briefly used to derive the derivatives of select functions, and then to move forward to the formal definition of the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. Afterwards, students move on to the different techniques for finding derivatives other than using the limit definition. I considered the idea of the difference quotient a 'simple thing', that is, I assumed that students had a solid understanding of the role it played in the development of the derivative, and I was concerned that the exploration I was proposing would not illuminate anything new for the students. Functions, derivatives, and the difference quotient can be graphed in Desmos easily so I hoped that this would allow the online students to share their screens and become a more active part of the class. My goal for this exploration was for the students to see visually that as we make h in the difference quotient smaller, the graphs of the difference quotient and derivative get closer. I did not expect the discussion to last more than a single class meeting, however by taking time to explore what I thought was a 'simple idea', students experienced surprises, and some misconceptions about limits were brought to light.

To begin the exploration of the difference quotient, students were asked to graph $f(x) = x^2$ using Desmos. Because one can graph the derivative in Desmos without needing to calculate the derivative, students were able to focus on the concept rather than the calculation of the derivative itself. Students then graphed $f'(x)$ and the difference quotient with an initial value of 4 for h. The initial Desmos graph can be seen in **Figure 1**.

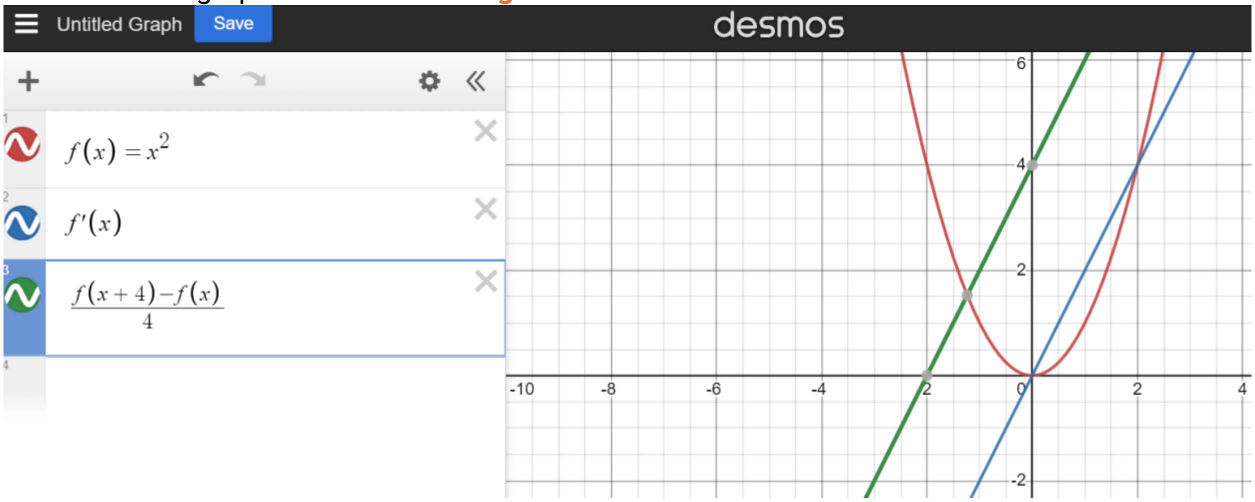

Figure 1. Initial Desmos graph.
Note: Illustrating the similarities of the graphs of the derivative of $f(x) = x^2$, and the associated difference quotient with $h = 4$.

A slider was available for h, but I wrote the activity to begin by using a value of 4 for h and asked students to change the value manually to begin the exploration because previous experience taught me that when students use the slider initially, they pull it back and forth quickly, make generalizations, but miss some of the finer details of what is happening. Slowing down this process at the beginning forces students to spend more time looking at how the graphs are changing. For example, by manually changing the value of h, students notice that the change in h appears to be directly proportional to the distance between the function $f(x) = x^2$ and the difference quotient along the x-axis. It was suggested to students that they hide the graph of the $f(x)$, and focus on what happened to the difference quotient as they change the value of h, making it smaller and larger. **Figure 2** shows the graphs of the derivative and difference quotient for three different values of h.

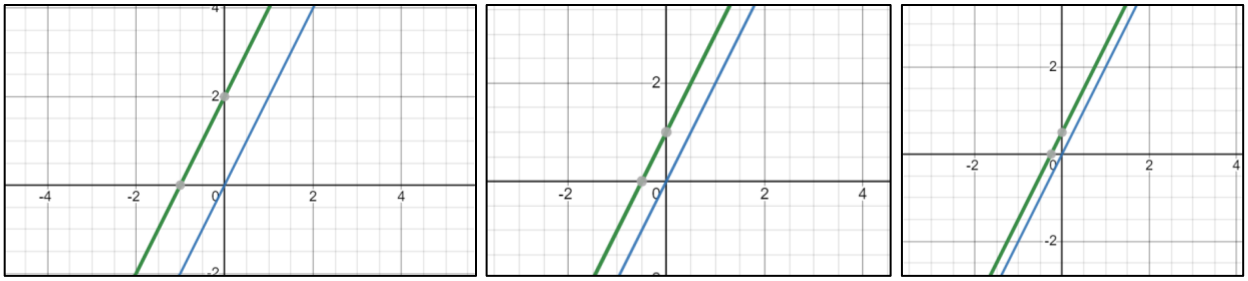

Figure 2. Changing h: $h = 2$ (left), $h = 1$ (center), $h = 0.5$ (right).
Note: The graph of the difference quotient moves closer to the graph of $f'(x)$ as h is decreased.

Initial Discussion

Students repeated this for the functions $f(x) = \sin(x)$, $f(x) = x^8 - x^4 + 2$, $f(x) = \sqrt{x}$, $f(x) = e^x$, $f(x) = 3$ and $f(x) = \frac{1}{x}$. Students worked together in pairs and recorded their thoughts on a worksheet (see **Appendix A**). Students noticed:

- For $f(x) = x^2$, the graphs of the derivative and the difference quotient appeared to be parallel, and the closer h was to 0, the closer the two lines. A student conjectured that the distance between the two lines was half of h.
- For most of the functions, students were confident that by making h small enough, the graphs of $f'(x)$ and $\frac{f(x+h)-f(x)}{h}$ became indistinguishable everywhere.
- It was noted that for the function $f(x) = 3$, both the derivative and the difference quotient were equal to 0.
- Something 'weird' was happening with the function $f(x) = \frac{1}{x}$. Whereas in the previous functions the graphs of the difference quotient and $f'(x)$ were similar in shape, the graph of the difference quotient had what a student called a 'weird U' in the middle which is seen in **Figure 3**.

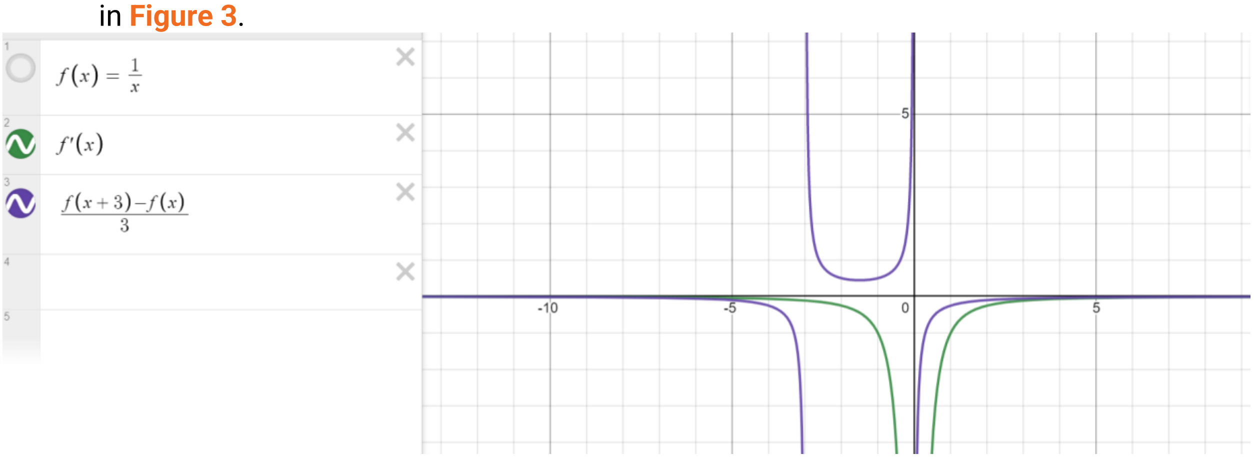

Figure 3. Graph with a 'Weird U'.
Note: The purple graph shows a 'weird U' that does not match the green graph of $f'(x)$.

Digging into Something Weird

The noticing of the 'weird U' in the middle of the graph of the difference quotient provided an opportune moment to sit and investigate why this was happening. Because students were using Desmos to graph the derivative and difference quotient, they needed to write these out for themselves. Once they had $f'(x) = \frac{-1}{x^2}$ and $\frac{f(x+h)-f(x)}{h} = \frac{-1}{x(x+h)}$, they were able to explain why the difference quotient had two vertical asymptotes, one at $x = -h$ and the other at $x = 0$ which was shared with the graph of the derivative. This was an unplanned review of asymptotes and a student wondered if the difference quotient would always have one more asymptote than the derivative.

This noticing of the extra asymptote in the difference quotient was troubling for many students. One student pointed out that no matter how small they made h, there would always be the extra asymptote. When the students had addressed this concept in Calculus I, they had accepted that $\lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$ by the rationalization that if h was small, then $x + h$ was essentially the same as x. However, now looking at the graphs, they were less willing to accept that claim. Struggling to make sense of what they were seeing, a frustrated student said that Calculus must have "a secret eraser" that comes and magically erases the asymptote as we calculate the limit.

Students appeared convinced that except for $f(x) = \frac{1}{x}$ and possibly other rational functions, the difference quotient was a 'good' approximation of the derivative, and one student made a joke that so much time had been wasted in Calculus learning to take derivatives when they could just use the difference quotient – this comment along with the unfinished discussion of the extra asymptote convinced me to continue the comparison of $f'(x)$ and the difference quotient at the next class meeting.

A Closer Look

Students were convinced that the derivative and the difference quotient were sufficiently close if h was small, and the function was not a rational function. The exploration that students had engaged in on the first day was like the quick exploration many Calculus textbooks provide students with. Given a function and a value for x, students intuitively determined the limit based on filling out a table (see **Figure 4**).

30. **[T]** Complete the following table for the function. Round your solutions to four decimal places.

x	***f*(*x*)**	**x**	***f*(*x*)**
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

31. What do your results in the preceding exercise indicate about the two-sided limit $\lim_{x \to 1} f(x)$? Explain your

Figure 4. Sample Calculus Problem (Herman et al., 2018).

One the second day, because the graphs of the derivative and the difference quotient were indistinguishable unless the function was a rational function, students were asked to examine the actual difference between the two graphs. Students used Desmos to graph the difference of $f'(x)$ and $\frac{f(x+h)-f(x)}{h}$ but this time hiding the original function graphs. Students worked in pairs and explored what happened as the value of h was changed for each of the functions and recorded their thoughts on a worksheet (see **Appendix B**).

There did not seem to be any surprises with $f(x) = x^2$. The graph in Desmos confirmed to the students that the derivative and difference quotient were parallel (something we could also have proven algebraically). The graphs for the other functions prompted interesting conversations and surprises.

It was noted that for $f(x) = \sin(x)$, the difference 'waved', and students took a few moments to make sense of that. The surprises began with $f(x) = x^8 - x^4 + 2$. Students noted that while the difference was small for x values near the origin, the difference 'grew out of control' the further away they moved, despite making h small as seen in **Figure 5**.

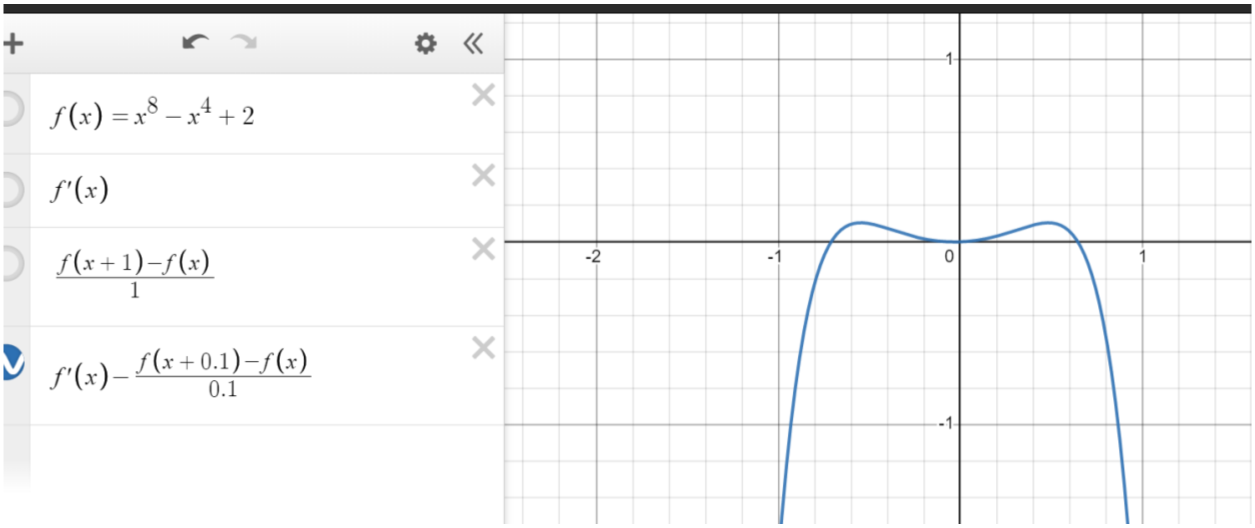

Figure 5. Out of Control Difference.
Note: Even with small h, the graph of the difference grows in the negative direction.

A similar phenomenon happened with the functions $f(x) = \sqrt{x}$, $f(x) = e^x$, and $f(x) = \frac{1}{x}$. There would be values of x for which the difference was small, but then would become large, something they had not been able to detect by their first experience with the graphs of the derivative and the difference quotient. Reexamining their previous work, students could see that even though the graphs of $f'(x)$ and the difference quotient may appear to be close in value, seen in **Figure 6**, graphing $f'(x) - \frac{f(x+h)-f(x)}{h}$ revealed a significant difference, seen in **Figure 7**. Some students expressed surprise that the two could be close for some values of x, but not for others. Because $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, students had anticipated that they could choose a small value for h such that the graphs of the derivative and difference quotient would be indistinguishable for all x. They wondered how small h had to be for the difference quotient to be a good approximation of the derivative for all values of x.

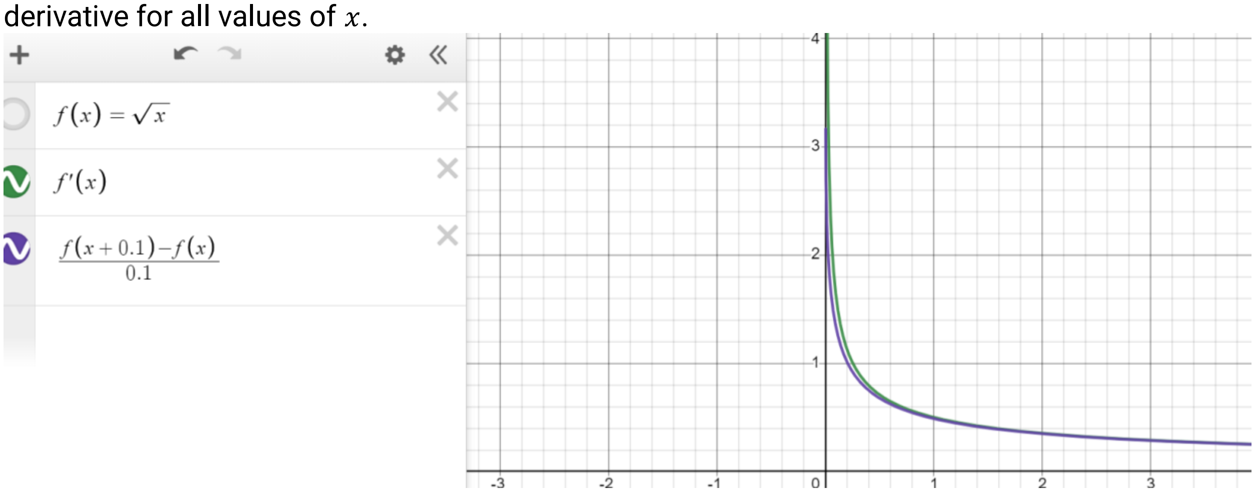

Figure 6. Apparently Similar Graphs.
Note: The graphs of the derivative and difference quotient appear to be close in value.

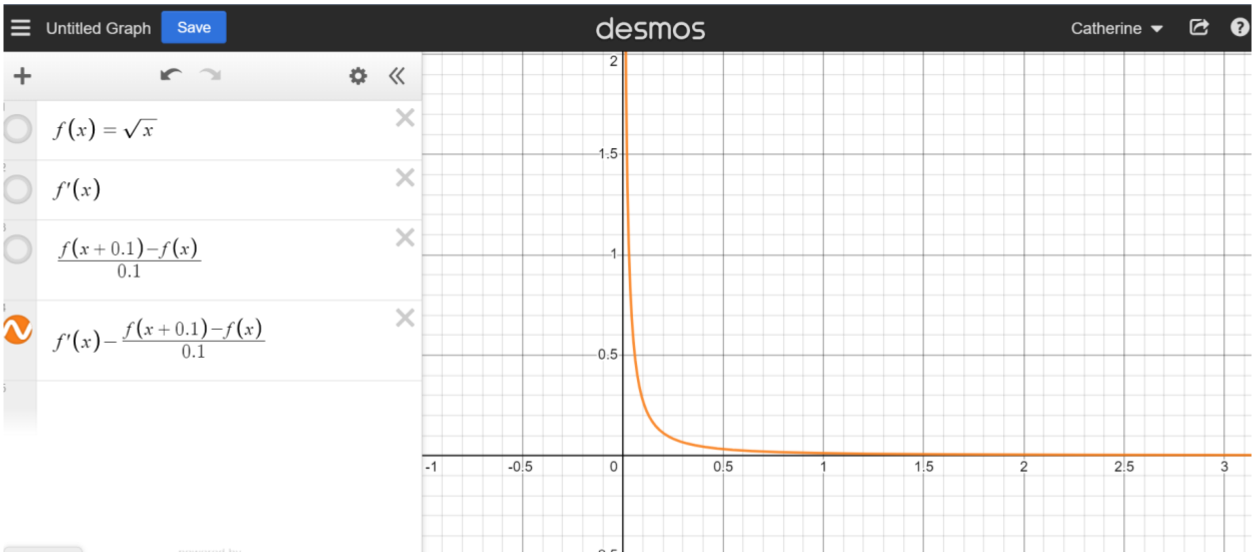

Figure 7. Difference Revealed.

Note: By graphing $f'(x)-\frac{f(x+0.1)-f(x)}{0.1}$, the actual difference is revealed.

The students had shifted their thinking from "the difference quotient is usually a good approximation" to "the difference quotient is a good approximation only in limited places". Working in pairs, students were asked to generalize where the difference quotient was a good approximation and where it was not. In the end, students settled on the idea that the 'steeper' the curves, the worse the difference quotient became at approximating the derivative.

Misconceptions About Limits Revealed

Students were now faced with two troubling ideas. First, they had not made sense of how the limit of the difference quotient for a rational function was equivalent to the derivative. How did taking the limit eliminate the extra asymptote? Second, even when the shape of the difference quotient matched the shape of the derivative, there were wide differences between the two, even for small values of h. As students voiced confusion, I realized that they had a misconception about, or an incomplete understanding of limits. They had previously believed that the difference quotient would move uniformly toward the derivative as h was decreased. After spending time examining the graphs of the difference quotient and the derivative, they now understood that even though the values of the two may be 'close' for some values of x, this did not necessarily mean that they were close for all values of x. Indeed, they may still be 'far' apart.

Moving Deeper

Returning to class on the third day, I had the goal of moving the conversation in such a direction that the formal definition of a limit could emerge. That is, could we use graphs in Desmos to help illustrate what we mean by $\lim_{x\to a} f(x)=L$ if for every number $\epsilon>0$ there is a number $\delta>0$ such that if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$? Students were asked if they could find a value of h for which the difference between $f'(x)$ and $\frac{f(x+h)-f(x)}{h}$ could be kept within different ranges defined by E (I chose to use E rather than ϵ on the worksheet – something I will change for future classes, as seen in **Appendix C**). Student were encouraged to use a slider for h. Horizontal lines helped visualize where the functions were outside the range of $(-E,E)$ as seen in **Figure 8**.

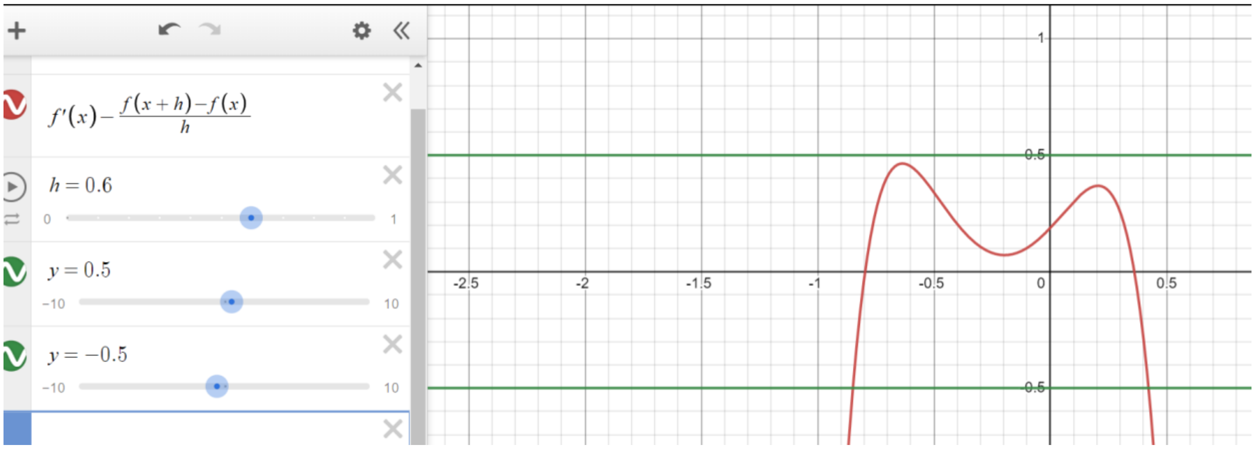

Figure 8. Capturing the Difference.
Note: The horizontal green lines help illustrate when the difference left the targeted area.

Students noted that except for the functions $f(x) = x^2$, $f(x) = \sin(x)$, and $f(x) = 3$, the functions all had at least one area in which the difference between the derivative and the difference quotient could not be 'controlled'. Students wanted to describe this by declaring that these other functions, such as $f(x) = e^x$, had places where the graph was so 'steep' that the difference quotient wasn't a good estimate. However, when pressed to define how 'steep' the function had to be for the difference quotient to fail to be a good approximation they ran into difficulties. At this point I made the decision to shift my original goal of discussing the formal concept of a limit in favor of pushing students to grapple with their idea that some graphs might be too steep to use the difference quotient as an estimate of the derivative.

Clarifying the Concept

I asked students to define what 'steep' meant to them. While there was some debate, all students agreed that if at some point a graph had a slope of $m = 1000$, then the graph could be considered 'steep'. To confront their idea that $f(x) = x^2$ was not 'steep', they were asked if it ever had a slope of 1000. Because the derivative is given by $2x$ they were quickly able to identify $x = 500$ as a place when the slope was therefore 1000, yet they had previously stated that the difference quotient for $f(x) = x^2$ was a good approximation. Therefore, since $f(x) = x^2$ had a 'steep' slope yet the difference quotient was still a good approximation of the derivative, it was not sufficient to say that for functions such as $f(x) = e^x$, the values for which the difference quotient varied from the derivative was determined only by the 'steepness' of the graph.

Using Desmos students explored other functions to make sense of this. They noted that for a cubic function, the graph of the difference between the derivative and the difference quotient was a line. At the suggestion of one student, they explored $f(x) = x^a$ for $2 < a < 3$. The graphs of these functions were a surprise to many of the students. Time was spent trying to predict the shape of the graph depending on the decimal. After much discussion and debate, students reached the conclusion that it was not the 'steepness' of the graph, it was how quickly the 'steepness' was changing that determined if the difference quotient could serve as a good approximation or not. Students tied in their previous knowledge of derivatives to this discussion noting that the derivative of quadratics is linear while the derivative of a cubic is a quadratic, therefore the graph of the cubic is not only steeper for many values of x, but the graph of the derivative is also changing at a faster rate further away from $x = 0$. We wrapped up the discussion by noticing that the ideas they were expressing could be expressed with the formal Calculus terminology they had learned in previous courses. Students had moments of laughter when they recalled that the second derivative provided information about how the first derivative was changing and this is exactly what they were seeing visually.

Spending Time with Other Topics

There were other concepts that I had only planned to discuss for a day, but which routinely expanded to fill a week or more based on the conversations and explorations they prompted in the students. The conversations that followed again indicated that time spent examining these concepts helped students develop a deeper understanding. For example, a discussion of the exponential function evolved into a competition in which the students were attempting to 'out run' the exponential with polynomials of ever-increasing power. A deeper look into Riemann Sums echoed the difference quotient discussion as students attempted to make sense of which functions could be adequately approximated by Riemann Sums and which could not. The discussion of limits reappeared as students were reminded that the definition of the area under a curve is defined as $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$, but the students were shocked by how many rectangles were needed for some functions in order to get a good approximation of the area under a curve. For these topics and others, I was often surprised by student comments which showed gaps in their previous understanding of the concepts.

Reflections

Even though I was unable to use many of the planned activities, the course goal of having students think about Calculus topics beyond algorithms and procedures was still met. The sudden shift to online explorations meant that much of the course was driven by student discussion – I had a rough plan of what I hoped students would discuss, but that was often set aside in favor of where students were steering the conversation. My fears that students would find some topics, such as the difference quotient, simple and not interesting were not realized. Following the student explorations, we stumbled into a series of challenging discussions that revealed some false ideas held by some students (i.e., that functions uniformly approach a limit). While we did not always get to a full discussion of topics I had planned, such as the formal definition of limits, I gained a better idea of how I might structure an exploration to reach that goal.

Student evaluations of the course were positive and indicated that the course goal of providing students with time to explore simple ideas deeply beyond algorithms and procedures was both met and beneficial to the students. On the course evaluation a student wrote "By not having to worry about if I was going to have to memorize specific content for a test, I felt as though I was actually learning more." As our future teachers, it is important for them to believe that it is beneficial (and possible) for students to engage with math for more than finding solutions to prescribed problems. I hope that the experience of having the time and freedom to explore Calculus topics will stay with the preservice teachers, and they will create similar opportunities for their own students, whether it be in Calculus, or some other math class. At the conclusion I was left wondering what other concepts I had been moving through quickly because I considered them simple, but which may have led to in depth conversations with more time. What activities might reveal misconceptions if I provide students with freedom to explore?

References

Association of Mathematics Teacher Educators [AMTE]. (2017). *Standards for Preparing Teachers of Mathematics.* Author.

Conference Board of the Mathematical Sciences [CBMS]. (2012). The Mathematical Education of Teachers II. American Mathematical Society and Mathematical Association of America.

Herman, E. & Strang, G. (2018). *Calculus Volume I.* OpenStax. Retrieved from https://openstax.org/details/books/calculus-volume-1

Jackson, A. (2001). Interview with Arnold Ross. *Notices of the AMS, 48*(7), 691-698.
National Council of Teachers of Mathematics [NCTM]. (2012). *NCTM CAEP Standards 2012.* Author.
Ohio Department of Education (2017). *Ohio's Learning Standards.* Retrieved from https://education.ohio.gov/Topics/Learning-in-Ohio/Mathematics/Ohio-s-Learning-Standards-in-Mathematics
Wilkerson, T. (2022, February). *Developing a Deep Understanding of Mathematics.* President's Message. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Trena-Wilkerson/Developing-a-Deep-Understanding-of-Mathematics/

Author Bio

Catherine Pullin Lane, *Baldwin Wallace University*, clane@bw.edu, Dr. Catherine Lane is an associate professor of mathematics and statistics at Baldwin Wallace University. She also serves on the Ohio Mathematics Education Leadership Council (OMELC). Dr. Lane's professional aspiration is to help all students find joy and creativity in mathematics.

Appendix A

Day One Worksheet

What's Happening?

Use the following table to record your observations and thoughts about what is happening to the difference quotient as we change the value of h for each of the following functions.

Function	Observations
$f(x) = \sin(x)$	
$f(x) = x^8 - x^4 + 2$	
$f(x) = \sqrt{x}$	
$f(x) = e^x$	
$f(x) = 3$	
$f(x) = \frac{1}{x}$	

Appendix B

Day Two Worksheet

What's Happening? – Part 2

As we did with $f(x) = x^2$, for each function in the table below, graph $f'(x) - \frac{f(x+h)-f(x)}{h}$. Begin with a value of 4 for h and then observe what happens to the graph as you change h. It may be easier to hide the graph of $f(x)$.
Use the following table to record your observations and thoughts about what is happening to the difference as we change the value of h for each of the following functions.

Function	Observations
$f(x) = \sin(x)$	
$f(x) = x^8 - x^4 + 2$	
$f(x) = \sqrt{x}$	
$f(x) = e^x$	
$f(x) = 3$	
$f(x) = \frac{1}{x}$	

Appendix C

Day Three Worksheet

What's Happening? – Part 3

As we did with $f(x) = x^2$, for each function in the table below, for each value of E, can you find a range of values of h such that $-E < f'(x) - \frac{f(x+)-f(x)}{h} < E$?

Use the following table to record your findings.

Function	$E = 1$	$E = 0.5$	$E = 0.1$
$f(x) = \sin(x)$			
$f(x) = x^8 - x^4 + 2$			
$f(x) = \sqrt{x}$			
$f(x) = e^x$			
$f(x) = 3$			
$f(x) = \frac{1}{x}$			