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KENTUCKY JOURNAL OF MATHEMATICS TEACHER EDUCATION

President's Message

On behalf of the Kentucky Association of Mathematics Teacher Educators (KAMTE), I hope you enjoy this issue of the *Kentucky Journal of Mathematics Teacher Education*. This publication provides a forum to build professional knowledge and exchange ideas in mathematics education and teacher preparation, and this mission is deeply aligned with the goals of KAMTE. Specifically, KAMTE aims to:

The purposes of KAMTE are:

1. To provide a vehicle for such purposes as addressing concerns, disseminating information and research, promoting effectiveness, and coordinating efforts in the preparation and continuing development of mathematics teachers.
2. To promote excellence in the preparation and continuing development of teachers of mathematics.
3. To advocate for high-quality mathematics education for all.
4. To establish collaborative working groups of mathematics teacher education professionals.

To achieve these goals, we have several activities planned for the coming year including free, online conferences for prospective teachers. The fall event will be held on November 3 (Friday), 2023 and another event planned for the spring (date TBD). We routinely engage in book clubs and other collegial activities focused on mathematics teaching and learning as well as our own professional goals, needs, and growth. More broadly, though, we aim to create community around shared interest mathematics education, and we very much hope you will consider joining our group.

What's Inside…

Finding Community in KAMTE

At this point, I think most everyone is aware of challenges regarding the recruitment and retention of mathematics teachers (and teachers in general). If you aren't, take a moment and do a quick Google search on the topic. A while back, I wrote an editorial (Thomas, 2019) that examined this shortage with respect to conditions, culture, and compensation. Had I the opportunity to revise that piece, I would add some thoughts on community. On the topic of community, Su (2020) writes, "This feeling – *I don't belong* – can be quite crippling. And this is a space where community is terribly important: for us to feel belonging . . . None of us can flourish without a supportive community – people with whom we share joys and sorrows, hopes and fears. A community helps us normalize struggle and realize, 'I am not alone in my struggle'" (p. 188). The field of mathematics education is complex, and the work before us can often seem overwhelming. Our first goal at KAMTE is to provide a place for each of us to share with one another, to commiserate with one another, to dream with one another, and to learn from one another. While this often materializes as conferences, meetings, book clubs, and other events, a key underlying value is that of community. Whether you are from Kentucky or from another state or country, *you are welcome in our community*. Whether you teach mathematics regularly, prepare mathematics teachers, or just enjoy spending time with the discipline, *you are welcome in our community*. Whether you are new to the education profession, or have a lifetime of experiences, *you are welcome in our community*. If you are reading this, *you are welcome in our community.*

Below are links to our organization's website and a membership link. In the spirit of community, I wholeheartedly invite you to peruse our website and get a sense of the group. If you have questions about what you see or read, do not hesitate to email me or any one of the KAMTE officers. We are proud of our community and love talking about it with others. Our hope is that you feel some connection with us and our mission and if you do, I encourage you to take the next step and click on that membership link. On behalf of the KAMTE organization, I very much hope that you will join our community and travel with us on our mathematics journey. I very much look forward to hearing from you.

KAMTE Website: https://kcm.nku.edu/KAMTE/index.php

KAMTE Membership Form: https://forms.office.com/r/C3jMa4bir4

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A Message from the Editors

Dear KJMTE Readers,

In this, the second issue of the *Kentucky Journal for Mathematics Teacher Education* (KJMTE), we are pleased to have two articles that address important issues for mathematics teacher educators to consider in their work with preservice teachers. First, Giang-Nguyen T. Nguyen presents procedures to help MTEs support preservice teachers in the development of problem-solving skills. Next, Kristy Litster and her colleagues share a framework to support preservice teachers' conceptualizations of interdisciplinary lesson planning. After reading both of these articles, I walked away with ideas to implement in my work with preservice teachers. I am confident you will too.

The KJMTE provides an open forum for both academic and informal discussions on various issues related to mathematics teacher education. As we publish more issues of KJTME, we hope to learn more ways to meet the needs of our readers. After publishing our first issue and preparing this second issue, we realize that there are more ways for writers to contribute to the journal besides a typical article. In future issues we introduce a new section of the journal titled "Commentary." Pieces published in the Commentary section of the journal will be of interest to teacher educators, but may not directly address the mission of the KJMTE to contribute "to building a professional knowledge base for mathematics teacher educators that stems from, develops, and strengthens practitioner knowledge." Commentary pieces are not peer-reviewed, but the editors will determine their appropriateness and will work with authors on the editing process.

Regardless of the type of publication, article or commentary, the journal will publish work which appeals to mathematics teacher educators – this includes mathematics educators, mathematicians, teacher leaders, school district mathematics experts, and others. We hope to encourage the development and sustenance of an equitable and welcoming environment for all individuals interested in mathematics education. If you are thinking about submitting an article for publication, please feel free to contact either of us to discuss your ideas. We would love to hear from you.

We have enjoyed putting this second issue together and we hope that you enjoy reading it. We look forward to getting your submissions and reading about the incredible work you do. Also, think about reviewing manuscripts for us. We need your input to make this journal meet the needs of mathematics teacher educators. Your reviews are vital to the success of KJMTE.

Finally, we hope that you find inspiration in this issue.

Bethany Noblitt, Ph.D. and Nicholas Fortune, Ph.D.
Co-Editors, KJMTE

AMTE Announcements

The 2022 AMTE Annual Report is available. In the annual report, you can read about the organization of AMTE and what great things AMTE accomplished throughout the year. You will learn about what AMTE has done related to publications, outreach, advocacy, and much more.

Did you know that AMTE has two podcasts available to listen to and learn from? The MTE Podcast accompanies the Mathematics Teacher Education Journal and the Teaching Math Teaching Podcast consists of conversations with mathematics teacher educators who are stepping into the role of teaching math teachers.

The 2023 AMTE Annual Conference will be held in Orlando, Florida, February 8-10, 2024. Registration is now available through November 30, 2023 with early registration at reduced rates available through September 30, 2023. Each year at the conference, there is an affiliate breakfast one morning of the conference. We would love to see you at the KAMTE table if you are attending the conference!

The AMTE Connections Newsletter for summer is available! The Summer 2023 newsletter includes a piece on developing preservice teachers' understanding and navigation of critical issues in teaching mathematics during their coursework.

Review for KJMTE

KJMTE is *your* journal. Reviewing articles for potential publication is a great way to have input into the types of articles KJMTE publishes for its readers.

The journal's aim is to provide a space for the exchange of ideas to advance mathematics teacher educator practice. Peer review of articles strengthens KJMTE's ability to meet this aim.

Interested in reviewing for KJMTE? Find out more at KJMTE.org.

Questions about KJMTE? Contact the KJMTE Editorial Team at editors@kjmte.org.

KAMTE Board Members

KAMTE would like to extend a warm welcome to our new board members. Dr. Jonathan Thomas, from the University of Kentucky, rejoins the KAMTE Board as our President-Elect. We are happy to have him back! KAMTE would also like to welcome our new At-Large Representatives, Dr. Michele Cudd from Morehead State University and Dr. Kate Marin from the University of Louisville. Dr. Marin also works with KAMTE social media. KAMTE is excited to have our board assembled and ready to support the mathematics teacher educators in Kentucky and beyond.

Jonathan Thomas, President

Jonathan Thomas is an Associate Professor of Mathematics Education and Chair of the Department of STEM Education at the University of Kentucky. Prior to his tenure at UK, he was a faculty member at Northern Kentucky University. Dr. Thomas is committed to a vision of STEM Education that is inclusive, engaging, and fosters a sense of relentless curiosity amongst students and teachers. He holds a B.A. in Elementary Education from the University of Kentucky, an M.Ed. in Educational Leadership and an Ed.D. in Mathematics Education, both from the University of Cincinnati. Dr. Thomas also serves as a faculty associate for the Kentucky Center for Mathematics (www.kentuckymathematics.org) and facilitates professional learning experiences for teachers across the commonwealth. Dr. Thomas has served as a mathematics intervention teacher in public, private, and charter schools in the greater Cincinnati metropolitan area. His research interests include investigating responsive mathematics teaching practices, equity concerns in the elementary mathematics classroom, non-verbal patterns of mathematical interaction, and cognitive progressions of children's mathematical construction.

Dee Crescitelli, President-Elect

Dr. Dee Crescitelli is a Director at the Kentucky Center for Mathematics and teaches as as adjunct at Georgetown College and the University of Louisville. She also serves as a Professional Learning Coach for Kentucky Adult Education. She is working to improve mathematics education from pre-K through college. Her teaching experience ranges from elementary through graduate school, adult education, and teacher preparation-threading real numeracy through all those levels.

Funda Gonulates, Past-President

Funda Gonulates is an Associate Professor of Mathematics Education at Northern Kentucky University and a faculty associate for the Kentucky Center for Mathematics. She received her Ph.D. from Michigan State University and is a former middle school mathematics teacher. She primarily teaches classes for elementary teacher candidates and elementary teachers. She worked on projects helping teachers build a classroom culture of mathematical sense-making. She is interested in creating a community of learners in a mathematics classroom and professional development settings. She works actively with Kentucky mathematics teacher leaders and aims to help them become change agents.

Jamie-Marie Miller, Secretary

Jamie-Marie Miller is an Assistant Professor in the Department of Teaching, Learning, and Educational Leadership at the Eastern Kentucky University. She received her Ph.D. from the University of Kentucky in STEM Education. Dr. Miller teaches elementary and middle/secondary mathematics methods courses, geometry for elementary teachers to undergraduates along with graduate courses in elementary mathematics education and intervention strategies for struggling learners. Her research focuses on the progression of algebraic thinking in students, math-specific literacy strategies, assessment, and visible learning practices.

Sue Peters, Treasurer

Susan Peters is an Associate Professor in the Department of Middle and Secondary Education at the University of Louisville, where she teaches mathematics methods courses and graduate courses in mathematics education. Her research focuses on statistics education and mathematics teacher knowledge, particularly teacher knowledge and education in statistics. When she's not working with teachers, she enjoys relaxing walks in nature.

Michele Cudd, At-Large Representative

Michele Cudd is an Assistant Professor in the Department of Early Childhood, Elementary and Special Education at Morehead State University, where she teaches future elementary, middle, and high school teachers. She is interested in supporting novice teachers to develop more student-centered discourse practices. In her free time, she often is hiking on trails with her dog.

Kate Marin, At-Large Representative

Kate Ariemma Marin is an Assistant Professor of Math Education at the University of Louisville. She has taught elementary and middle school and served as a math coordinator in schools across Massachusetts. Prior to the University of Louisville, she was a faculty member at Stonehill College. She teaches mathematics education courses and supports the development of pre-service and in-service teachers. Her research interest is in teachers' development of Mathematical Knowledge for Teaching and generational differences in teachers. She is committed to supporting teachers and promoting the knowledge that they bring to the profession.

KAMTE Membership

Membership to the Kentucky Association of Mathematics Teacher Educators (KAMTE) is always open for any faculty member that works with preparing pre-service and in-service teachers at any level. To join, visit KAMTE Website at https://kcm.nku.edu/KAMTE/index.php, access the KAMTE Membership Form at https://forms.office.com/r/C3jMa4bir4 or contact Treasurer Sue Peters at s.peters@louisville.edu.

Upcoming Conferences

Oct. 25-28, 2023	NCTM Annual Conference	Washington, DC
Oct. 24-25, 2023	NCTM Research Conference	Washington, DC
February 8-10, 2024	Annual AMTE Conference	Orlando, FL
March 4-5, 2024	KCM Conference	Lexington, KY

Call for Manuscripts

The editors of KJMTE are soliciting manuscripts for publication in the next issue of *the Kentucky Journal of Mathematics Teacher Education* that builds on the theme of the first issue: “The Next Generation of Mathematics Teachers.”

Specifically, we ask authors to consider the following: What are the next generation of mathematics teachers? What are their needs? What role do mathematics teacher educators have in meeting those needs? How can mathematics teacher educators best prepare the next generation of mathematics teachers for their work?

The journal’s aim is to provide a space for the exchange of ideas to advance mathematics teacher educator practice. The journal welcomes manuscripts that support this aim. Of particular interest are manuscripts that address an issue in mathematics teacher education and the methods/intervention/tools that were used to investigate the issue along with the means by which results were determined and the impacts on practice. Manuscripts should fall into one of the following categories:

Manuscripts that describe effective ways of influencing teachers’ knowledge, practice, or beliefs. This might include a description of activities, tasks, or materials that are used by a teacher educator to influence teachers in some way. These manuscripts would include a rationale for the intervention, a careful description of the intervention, discussion of the impact of the intervention, and how it might be used by others.

Manuscripts that describe the use of broadly applicable tools and frameworks in mathematics teacher education. This might include a classroom observation protocol, a task analysis framework, assessment tasks, or a framework for a teacher education program. These manuscripts would include a careful description of the tool or framework, what it is designed to capture, its use, and a discussion of the outcomes. The manuscript should include an explanation of how to interpret the results of the data captured by the tool. The tool should be made available for other professionals to use, modify, enhance, and study.

If you are interested in writing a manuscript for an issue of KJMTE, please visit the KJMTE Current Call for Manuscripts for the Author Toolkit where you can find formatting guidelines and information for preparing and submitting a manuscript to KJMTE.

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Solving $a \sim b$: Where Mathematics Teacher Educators' Expectations and Students' Experiences Meet

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Abstract

The author presents how to support preservice teachers (PSTs) in the development of problem-solving skills utilizing the following procedures: (1) *assess* PSTs' knowledge levels of problem-solving by utilizing a specified task; (2) *examine* PSTs' varying solutions to the selected task; (3) *discuss* PSTs' needs in developing and supporting problem-solving skills; and (4) *identify* the role mathematics teacher educators (MTEs) play in meeting PSTs' needs. The author ends with implications on how MTEs may best prepare the next generation of mathematics teachers.

Keywords: Problem-Solving, Non-Routine Problems, Mathematics Teacher Educators

Problem-solving appears in mathematics education curricula worldwide (Mwei, 2017). In the United States, the National Council of Teachers of Mathematics [NCTM] (1989; 2020) has continued to emphasize problem-solving as an important mathematical process for multiple decades. The problem-solving process involves problem-solvers using skills creatively in new situations (Aydogdu & Ayaz, 2008). According to Faradillah et al. (2018), K-12 students are expected to solve non-routine problems to develop problem-solving skills, but they are provided with routine problems. They also indicated students might feel familiar with non-routine problems because of limited exposure to these kinds of tasks and stated that solving non-routine problems increases students' mathematical reasoning. Also, students are likely to solve and excel at routine problems, but they are unlikely to solve non-routine problems, so they have limited problem-solving strategies (Or & Bal, 2023). Non-routine problems require problem solvers to use more than just applying learned procedures to solve the problems. Even if the path to a solution is unknown to students and they might be unfamiliar with non-routine problems, these problems could "encourage logical thinking, add conceptual understanding, develop mathematical reasoning, develop abstractive thinking skills and transfer math skills to unfamiliar situations" (Faradillah et al., 2018, p. 3). A question for consideration is how to promote the inclusion of such non-routine problems to K-12 students to help them develop their problem-solving skills. One can see that mathematics teachers are the mediators of this integration, so they must possess the knowledge to teach problem-solving skills. Accordingly, they should experience problem-solving in a manner similar to what we would like their students to demonstrate (Rigelman, 2007). Faradillah et al. (2018) suggested that PSTs must possess the knowledge and ability to solve these problems and that they should receive such preparation as pre-service teachers (PSTs).

To better understand PSTs' knowledge and ability to solve non-routine problems, their knowledge must be assessed. A previous study (i.e., Wilburne, 2006) found PSTs do not have the aforementioned knowledge to teach their students; thus, they were not prepared. Specifically, PSTs were limited in their problem-solving approach; they "rarely plan and follow procedures when solving problems" (Mataka et al., 2014, p. 173). However, Barham (2020) indicated that teaching PSTs about problem-solving approaches, for example Polya's problem-solving approach (1957) would support their development.

Polya (1957) discussed four problem-solving principles: understand the problem, devise a plan, carry out the plan, and look back. Polya (1969) indicated that one main point of mathematics teaching is to develop the tactics of problem-solving, so supporting PSTs' problem-solving skills or new knowledge development may also be implemented in their preparations (Ebby, 2000). Similarly, NCTM (2014) suggested that PSTs should be engaged in solving "challenging tasks that involve active meaning making and support meaningful learning" (p. 9).

In developing PSTs' problem-solving skills, it is important to give them authentic learning experiences in their preparation, the experience that Schoenfeld (2016) referred to as "mathematizing" (p. 17), and mathematics teacher educators (MTEs) could do so by providing PSTs with different problem-solving experiences (e.g., solving non-routine problems). In this paper, the author provides ideas on how MTEs could support PSTs with problem-solving skills in a mathematics methods course through a process shown in Figure 1. PST's knowledge of problem-solving skills was assessed, solutions to the task were examined, needs were discussed, and MTEs' roles in meeting their needs were identified.

Figure 1. Process for Supporting the Development of PSTs' Problem-Solving Skills.

Assess Pre-Service Teachers' Knowledge of Problem-Solving

To gain insights into PSTs' experience with problem-solving, the MTE provided a non-routine task shown below. The selected task has different features that require PSTs to use the knowledge they have learned to find the solution. PSTs' knowledge was assessed through a task adapted from the Mathematical Sophistication Instrument (Szydlik et al., 2013).

The Original Task:
The notation $a \sim b$ means multiply together a copies of b then add 1. For example, $3 \sim 2 = 9$. Which of the following is equivalent to 25? (a) $2 \sim 5$; (b) $5 \sim 2$, (c) $3 \sim 8$; (d) None of the above.

The Adapted Task:
$a \sim b$ means multiply together a copies of b, then add 1. For example, $3 \sim 2 = 9$. Find a and b that satisfy $a \sim b = 25$?

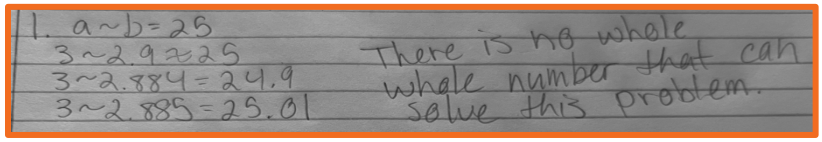
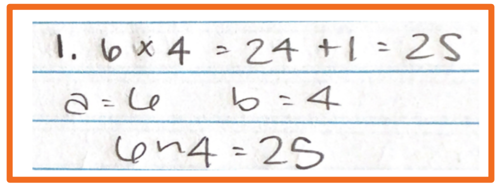
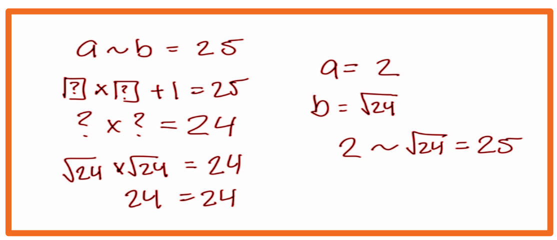
The task was assigned to PSTs enrolled in an elementary mathematics methods course at the university. These PSTs are juniors or seniors in the elementary education and/or exceptional student education program; they were required to take one mathematics methods course to fulfill the initial teaching certificate. PSTs completed the task and uploaded their solutions to the course Canvas Shell prior to class. The MTE examined these solutions to the task to prepare for a face-to-face discussion with PSTs. PSTs' solutions were grouped into five categories for the selected task. These solutions set were examined to learn more about their needs.

Examining PSTs' Varied Solutions to the Selected Task

Various solutions were reported for the task collected from 25 PSTs. In Table 1, there are samples of PSTs' work and their justifications for the value of a and b.

Table 1. PSTs' Answers, Explanations, and Work on Selected Task.

#	Student Answer	Explanation and Work
1	$a = 2, b = 5$	My thought process is $5 \sim 2$ because 3 to the power of 2 is 9, and my guess is that is the way to solve this type of problem.

2	$a = 3, b = 2.885$	
3	$a = 6, b = 4$	
4	$a = 1, b = 24$	I rewrote the equation to $b^a + 1 = x$, as this was a better reminder of the notation $a \sim b$. I then plugged in the known number $b^a + 1 = 25$ and solved what I was able to, subtract 1 from each side of the equation so $b^a = 24$. From here I was not able to remember how to solve for both a and b in this equation and instead plugged in different numbers; if $b = 1,2,3,4 \ldots$ then is there a whole number for a that would solve the problem?
5	$a = 2, b = \sqrt{24}$	

PSTs' solutions to the selected task were examined to determine their needs in their development of problem-solving skills. The MTE planned for the class to discuss the solution to the problem through some selected answers, whether the answer was correct or not, and discuss some pedagogical considerations in teaching students through problem-solving.

Discuss PSTs' Needs and Identify MTE's Roles

In the class meeting, varying answers were reported on the dry-erase board, and the MTE selected the incorrect answers to discuss first, as listed above. This *selecting* approach of the five practices (Stein et al., 2008) allowed PSTs to see how an instructional strategy may orchestrate discussions within classrooms (Nabb et al., 2018). The MTE emphasized how some specific solutions were selected, and PSTs were asked to provide justifications for their answers to develop PSTs' knowledge levels. In examining PSTs' solutions, the following outcomes were observed: (a) PSTs were not familiar with solving non-routine problems, (b) PSTs did not read all given information, (c) PSTs tried to remember where they learned about the sign "~", and (d) PSTs exhibited limited problem-solving strategies. Observations gained through the examination of task answers provided specific PSTs' needs and MTEs' roles as summarized in Table 2 and discussed in detail following Table 2.

Table 2. The Needs and Roles.

#	PSTs' Needs	Mathematics Teacher Educators' Roles
1	Experience with solving non-routine problems	Selecting and adapting tasks that promote the development of non-routine problem solving
2	Problem-solving strategies	Present Polya's approach to problem-solving
3	Limited knowledge about teaching mathematics through problem-solving	Co-construct the role: Discuss pedagogical approach for teaching mathematics through problem-solving

Need 1: PSTs Limited Experience with Solving Non-Routine Problems

PSTs tend to be able to solve routine problems, so when it comes to solving non-routine problems, they were having a difficult time, just like those documented in Dündar & Yaman (2015). But some PSTs found the answer to the problem with procedural knowledge using a problem-solving approach. Some PSTs did not know what to do with the presented problem (one stated, "I got frustrated and didn't know what to do") and tried to find what $a \sim b$ means. Other PSTs shared their experiences solving this task:

- I don't know what "a copies of b means," so I tried to look in my mathematics books.
- I looked up "$a \sim b$" but could not locate information from any math books.
- My spouse is an engineering student…and didn't know what "$a \sim b$" is.

These PSTs likely did not read the information provided in the problem. Other PSTs shared that they started unpacking the problem by trying to make sense of the given information, "a copies of b plus 1." PSTs did not know what this statement meant at first; however, some of the PSTs worked through the provided example of $3 \sim 2 = 9$ and arrived at the answer, $(2 \times 2 \times 2) + 1 = 9$.

After figuring out the answer to $3 \sim 2 = 9$, PSTs indicated working backward, beginning with 25, and subtracting 1 to get 24. Then, these PSTs looked at different whole number factors of 24 (1 and 24; 2 and 12; 3 and 8; 4 and 6). Some PSTs were confused between exponents and multiplication and arrived at different answers (*See Explanation 3*). There was a lively discussion of the $1 \sim 24$ solution. Some PSTs argued using the following logic: if they followed what was given to them in the problem, $1 \sim 24$ means "multiplying 1 copies of 24, $1 \sim 24$." However, this reasoning does not make sense. PSTs showed their frustration with the solution, yet they had not reached the conclusion that they were solving $b^a + 1 = 25$. According to Bloom (2007), students may easily become frustrated when solving problems; however, with appropriate scaffolding, students begin to think in abstract terms about the mathematics used to solve problems. This promotes effective PST skills, so the MTE must select and adapt tasks (e.g., non-routine problems) for PSTs to solve so they become familiar with these types of problems.

Role 1: Selecting and Adapting Tasks for Promoting the Development of Problem-Solving Skills

As discussed in **Need 1**, the adapted task provided PSTs with experience solving non-routine problems. The MTE was strategic in choosing tasks to help PSTs think flexibly about teaching mathematics. The task fostered conversations related to how PSTs addressed the problem. PSTs shared that the task was unfamiliar to them; therefore, they struggled to find the answer. MTEs should try engaging PSTs to have a conversation about choosing a routine problem vs. a non-routine problem for use with their future students. If the consideration and setting are plausible for incorporating an adapted task, MTEs may elect to discuss with PSTs about using Polya's (1957) approach to solve the problem. If the MTE had used the original task with multiple-choice answers, then PSTs may have chosen the correct answer without much thought or as a random choice. However, the MTE modified the task with the aim of eliciting rich discussion for promoting classroom discourse (Calor et al., 2020). The modified task assisted PSTs in thinking about the question differently. Moreover, discussing the task features and/or its cognitive demands (Henningsen & Stein, 1997) promoted problem-solving skills and developed strategic competence (Kilpatrick et al., 2001). Through scaffolding, PSTs thought deeply about the mathematics they used in solving the problem (Bloom, 2007); therefore, the MTE played a role in developing PSTs' problem-solving skills through the tasks they selected for engaging PSTs. Thus, PSTs need to be taught different problem-solving strategies.

Need 2: Problem-Solving Strategies

When asking PSTs to share their experience with problem-solving, they shared they have limited knowledge about problem-solving strategies; as shown in this study, they do now know how to ask questions to help them better understand what the question was asking, which was consistent with prior research studies (e.g., Barham, 2020). For example, as shown in *Explanation 1*, the PST did not use the information provided, $3 \sim 2 = 9$, to find $2^3 + 1 = 9$. Rather, the PST used the fact that $3^2 = 9$ to reach the conclusion that the answer is $5^2 = 25$, leaving the information "plus 1" out in the solution strategy. Reflecting upon the experience, the PST stated that she did not try to understand what the question asked. Her reasons for how she arrived at the answer are similar to *Explanation* 1 (shown previously): that it is $5 \sim 2$ because she used the same approach $3 \sim 2$ to get 9. She reflected,

> The first step was to look at the example problem, which was $3{\sim}2 = 9$. When looking at the numbers and the representation of the problem, I then viewed this problem as doubles, meaning how many times 3 can be multiplied to get to 9. This can be twice, meaning $3^2 = 9$. From this, one can determine that $5{\sim}2 = 25$ because 5×5 or 5^2 is equal to 25.

A more helpful question is, "How does $3 \sim 2 = 9$?" To help PSTs, the MTE posed the question, "What does multiply together 3 copies of 2 mean to you?" The posed question helped PSTs realize they needed to figure why $3 \sim 2 = 9$. One PST made the following comment: "My assumption is that b could be in a square root!" What the PST meant was that b could be a non-integer number. In the end, most PSTs concluded $a = 2$ and $b = \sqrt{24}$ is the best answer (as shown in *Explanation 5)*. However, PSTs also indicated $1 \sim 24$ would be more appropriate for elementary students. Based on this need, MTEs must teach PSTs about approaches to problem-solving.

Role 2: Teaching About Problem-Solving Strategies – Present Approach to Problem-Solving

As shown in Need 2, many PSTs did not try to understand the problem. If PSTs are not familiar with Polya's (1957) approaches to problem-solving or have forgotten these approaches, then the MTE should present Polya's approaches and/or review this information. The MTE challenged the PSTs with the task and at the same time taught them about problem-solving skills. The MTE reminded PSTs of Polya's approach to problem-solving and modeled how to solve the problem through the four steps:

Step 1: Understand the Problem

In order to solve $a \sim b = 25$, one needs to know what $a \sim b$ means. The information indicates $a \sim b$ means multiply "a copies of b"; therefore, one needs to examine why $3 \sim 2 = 9$, multiply 3 copies of 2, then plus 1, which is $(2 \times 2 \times 2) + 1$, to get 9.

Step 2: Devise a Plan

$a \sim b = 25$. PSTs need to work backward: subtract 1 from 25, then find $b^a = 24$.

Step 3: Carry Out the Plan

Subtract 1 from 25: the result is 24. Now find a copies of b to result in the value of 24 (i.e., $b^a = 24$). PSTs tried different combinations of numbers.

Step 4: Look Back

While working on Step 3, PSTs would sometimes check to see if their answers made sense and rework the problem until a solution was reached. In *Explanation 2*, the PST made the following assumption: "There is no whole number that works for this question. On the other hand, in the solution presented in *Explanation 4,* the PST suggested that 1 could be a value for a, but while PSTs were checking the wording in the context of the problem, "multiplying 1 copy of 24" did not "sound right," as discussed in **Role 2**. Also, if the assumption was a should be a number greater than 1 and b could be a non-integer number, in this case, the number is a radical number and *Explanation* 5 is the best choice. Furthermore, a discussion of a non-integer value of b, as shown in *Explanation 5,* provided a good opportunity for all PSTs to engage in a discussion leading to a potential conclusion and finding as follows: $b^a + 1 = 25$.

Need 3: Limited Knowledge in Teaching Mathematics Through Problem-Solving

As part of the mathematics methods course, PSTs had opportunities to discuss approaches to teaching mathematics concepts to elementary students. In addition to the authentic learning topics and skills with problem-solving, PSTs were asked to share their thoughts on some pedagogical considerations, such as the following: (1) Where in the elementary curriculum is problem-solving, as represented by these illustrated tasks, appropriate for inclusion? (2) How will you assist your future students to solve these types of problems? and (3) Why is the selected task a good or bad task for elementary students? Discussions helped PSTs realize how designing, selecting, adapting a task could foster their future students' mathematical fluency. In solving, $a \sim b$, the following aspects of mathematical proficiencies (Kilpatrick et. al, 2001) were presented: conceptual understanding (transfer of knowledge and apply the knowledge for solving $3 \sim 2 = 9$), procedural fluency (carrying out procedures for finding $a \sim b = 25$ with flexibilities), and strategic competence (formulate and solve the problem $a \sim b = 25$). PSTs were pushed to think about what they know, how facts and methods learned with understanding are connected, and how those facts and skills were easier to remember and use or how they can be reconstructed when forgotten. As a result, PSTs seemed to better understand the methods used to create conclusions, and PSTs are more likely to apply this process in their future teaching (Ebby, 2000). For example, a PST whose work is in *Explanation 2* shared the following reflective discussion:

> To help students solve the task … I would break it up step by step and do it as a class. I still haven't fully found the answer to this question, but I think with help from peers and guidance I would be able to. Doing this as a whole group asking different students to try different numbers will allow the class as a whole to put their brains together and hopefully find the right answer.

Her response indicated that the PST had limited knowledge on how to solve the problem herself which suggested she might have limited knowledge about teaching problem-solving to her future students as suggested by previous research (e.g., Faradillah et al., 2018). Similarly, another PST shared, "I am not sure how I would help students, because I required help solving it too…" Hence, it was important for the MTE to discuss the pedagogical considerations in teaching through problem-solving.

Role 3: Co-Construct the Role – Discuss the Pedagogical Approach for Teaching Mathematics Through Problem-Solving

In discussing approaches to teaching through problem-solving, the evidence shared in **Need 3** strongly suggests MTEs and PSTs co-construct the role, that is, the MTE would be an active listener and PSTs would have opportunities to share their experiences in solving the task and to consider how the task could be used with their future students. Also, the MTE could create an opportunity for PSTs to think about teaching mathematics through problem-solving in their future classroom by focusing on the following considerations.

Promoting the Originality of Students' Work

The MTE modeled how to promote the originality of each group member's post; a PST solved the problem and submitted the response. One cannot see their group members' responses until the discussion. Also, the MTE could anticipate questions from PSTs. Below is an excerpt from a conversation between the MTE and a PST:

PST: I understand how to work number 1 ($3 \sim 2 = 1$), but with number 2 having an answer of 24, I can't figure it out. Can you please assist me?
MTE: Hello…. How did you get the answer for $3 \sim 2 = 9$? Use the same approach and see if you can figure it out.
PST: I did $2 \times 2 \times 2$ and got 8. Then I added one to eight and got nine.
MTE: Have you tried to work backward with 25 and see if you can come up with the answer?
PST: Would it be $5{\sim}2 = 25$?
MTE: If that is the case, then would $3{\sim}2$ equal 9? Try to go back to the first algorithm that you did and see how you would get 9. Maybe you can discuss this with your group members and figure this out. I encourage you to give it a try. You are getting there.
PST: Okay, I got it figured out. Thank you so much for your help.

In the above conversation, the MTE must decide when to give PSTs cues (i.e., prompts) (Hoffman & Spatariu, 2008). The MTE could provide a strong prompt and give away the answer, or a weak prompt, to make the PSTs think more about the problem. Instead of answering, the MTE asked, "How did you get 9?" suggesting that the PST use a similar approach to find $a \sim b = 25$. The MTE asked questions without disclosing to the PST that "a copies of b" is b^a.

Supporting PSTs as they develop mathematical problem-solving knowledge for teaching in elementary schools is vitally important (Barham, 2020). PSTs need to experience problem-solving to become better prepared to teach about problem-solving, and MTEs must set the tone in all classroom discussions.

Setting the Tone: Making Students Share their Experience Solving Tasks

MTEs should learn about PSTs to better support their educational journey and to assist PSTs to share their problem-solving experiences. Most importantly, MTE must set the stage for class discussions at the beginning of the semester. Earlier in the course, the MTE emphasized to PSTs how using open tasks would promote higher-level thinking; for example, 'find the sum of 3 and 4' vs. 'find two numbers that have the sum of 7' (Tran & Nguyen, 2021). When PSTs were asked to solve such open tasks, they reflected their preferences were to solve higher cognitive demanding tasks (Henningsen & Stein, 1997). After solving a mathematical task, the class would discuss features of the task aimed at supporting mathematical proficiencies (Kilpatrick et al., 2001). For example, after PSTs solved the task, $a \sim b$, they were asked if the problem is appropriate for elementary students. One PST shared the following commentary:

> I think this problem is appropriate for a 4th grade class. After looking at the number and operations and algebraic thinking standards, I found that it would best fit in the 4th grade. Students are learning how to multiply with automaticity and that helps when dealing with exponents in my opinion. Once you have that automaticity, it will be easier.

Some PSTs indicated "No, this would be too difficult!" At that time, the MTE prompted: "If elementary students are given a number and asked to multiply that number three times, then add one, could students solve that task?" PSTs answered "yes," and they agreed the present task is an elementary mathematics problem but suggested they would use the phrase "b is multiplied by itself a times" so it is more developmentally appropriate. The comment about the problem's appropriateness for 4th grade students could also generate a good discussion. The MTE could extend the discussion to ask the PSTs why it is appropriate for 4th grade by asking the class to access the state standard curriculum to validate. In this study's state, this task aligned with the state standard for the Algebraic Reasoning Strand: "Generate, describe and extend a numerical pattern that follows a given rule." (Florida DOE, https://cpalms.org/public/search/Standard). The PSTs' response allowed the following reflection on teaching: the MTE co-constructed the role, whereby the PST had a chance to think about their future teaching. Through the case of solving for $a \sim b$, the needs and MTEs' roles were discussed, revealing a meeting point between PSTs' experience and MTEs' roles.

Finding the Intersection Between MTS' Expectation and PSTs' Experience

For the task $a \sim b$, perhaps the MTE had expected PSTs to work backward to make sense of the given information, $3 \sim 2 = 9$. The MTE would expect PSTs to realize the correct strategy for solving the problem was $2 \times 2 \times 2$ plus 1. However, if the PSTs did not see the answer or use this approach, the MTE must provide informative discussion for PSTs on how they can solve the problem. Additionally, the MTE anticipated PSTs would "transfer their knowledge" learned from $3 \sim 2 = 9$ to solve $a \sim b = 25$. Even if the PSTs' experience with problem-solving was not to the level MTEs expected, MTEs should accept PSTs where they are mathematically and logically and assist them to develop mathematical proficiencies to help their future students. MTEs must find an intersection between their expectations and PSTs' needs to help PSTs enhance their "problem-solving abilities and altitudes" to move forward (Wilburne, 2006, p. 462). MTEs must furthermore attend to individual needs, as the one presented here, where the PST was not able to find the solution:

> I cannot seem to find the answer to this problem. Looking at it, I thought it would be $2 \sim 5$ because that answer would be 25. However, I forgot that you have to add one. I keep plugging in different numbers and even decimals and I cannot solve it.

Implications for MTEs

As indicated by NCTM (2014), engaging in challenging tasks results in mathematics learning. Therefore, MTEs should provide an opportunity for PSTs to solve problems. Continuing a conversation on how MTEs could assist PSTs in developing problem-solving skills is also vitally important. Particularly, MTEs must attend to PSTs' needs and provide appropriate time for PSTs to grapple with non-routine problems and to build PSTs' skills to stimulate problem-solving, as well as ask purposeful questions about teaching problem-solving to their future students. Teachers must incorporate non-routine problems into the existing curriculum. In this study, PSTs worked on the task outside of class and utilized class time for pedagogical discussions. An MTE does not have control over the knowledge or experience PSTs bring to classrooms, but, as suggested by Mataka et al. (2014), MTEs should acknowledge PSTs' limited experience to support their development of problem-solving skills. Considerations arising from the case of solving $a \sim b$ has prompted the author to suggest some pedagogical considerations MTEs should consider in designing courses for preparing PSTs:

(1) MTEs could attend to PSTs' needs by acknowledging or retrieving evidence concerning PSTs' prior experiences with problem-solving. What can be done to prevent PSTs from being placed in classrooms without knowledge of problem-solving skills? As presented, the MTE implemented the task to assess PSTs' skills and at the same time assist PSTs to develop mathematical knowledge for teaching problem-solving. When a PST shared, "I did not remember learning $a \sim b$ in school and tried to find information in their own textbooks," the MTE acknowledged PSTs may have not been exposed to such problems and explained to them what problem-solving entails (Ebby, 2000).

(2) MTEs must provide PSTs with authentic learning experiences in problem-solving. PSTs come to teacher education programs with limited experience in problem-solving, demonstrated by their solution approaches to the task, $a \sim b$. PSTs exhibited weak knowledge in applying "essential skills required for success in solving mathematical problems" (Barham, 2020, p.139). MTEs have the responsibility to develop PSTs problem-solving skills and knowledge through authentic learning experiences as problem solvers. As suggested by Ebby (2000), methods courses should provide "new learning experiences that challenge preservice teachers' beliefs about teaching and learning mathematics" (p. 95).

(3) MTE must support PSTs in their development of problem-solving skills. Developing PSTs' problem-solving skills as learners to prepare them for their future classroom challenges is critical work for MTEs. By diagnosing PSTs' knowledge, experience, and their thinking toward teaching mathematics, MTEs may realize more of the roles they play in preparing PSTs to complete their preparation programs (Ngcobo, 2021). PSTs' development of problem-solving skills is a necessity for promoting success in the teaching and learning of mathematics.

Conclusion

With a strong emphasis on K-12 students learning mathematics through problem-solving, the next generation of mathematics teachers must be well prepared to teach these students: these teachers should be ready for the important work. Previous research (e.g., Rigelman, 2007) suggests PSTs should experience problem-solving in a manner similar to what their students can demonstrate, and in this paper, the author provided an example of how it could be done in a mathematics methods course. Problem-solving requires PSTs to apply the knowledge they learned and translate it to a new problem. This knowledge of problem-solving skills is specialized content knowledge (Ball et al., 2008), and it is important for PSTs to possess this knowledge prior to completing their teacher preparation program (Ngcobo, 2021). Therefore, MTEs should find tasks that foster the development of problem-solving ability for them to reason and communicate mathematically (NCTM, 1991) as they engage in problem-solving tasks, for example, solving non-routine problems. PSTs' knowledge and skills can be learned from their solution approaches as well as their pedagogical considerations to help them develop the knowledge for teaching. Here, PSTs solved the one problem of $a \sim b$ and by examining the solution to the problem, MTEs learned so much about their needs. PSTs' needs were revealed and MTEs' roles in helping their development were discussed. There is more to learn about PSTs' needs to better support them, so more research with empirical data related to PSTs' experiences with problem-solving would provide insights into how to support their development.

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Exploring the Purposes of Interdisciplinary Connections in Pre-Service Elementary Teachers' Mathematics Lessons

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Abstract

This paper brings forward the Purposes of Mathematics Integration framework as a tool to support elementary pre-service teachers' (PSTs') conceptualizations of interdisciplinary lesson planning. The tool explores two interdisciplinary trajectories: level of integration and organization. PSTs' use of these two trajectories supports four interdisciplinary lesson planning purposes: 1) focus on math for the sake of math, 2) situate the relevance of math, 3) explore relationships between math and other content areas, and 4) explore authentic applications of math. This article also discusses how this tool was used to evaluate outcomes relating to 47 PSTs' initial conceptualizations of the interdisciplinary lesson plan and instruction focusing on math and another content area. Findings show that comparisons of the four purposes within teacher education programs can increase interdisciplinary connections in PSTs' elementary math lesson plans.

Keywords: Elementary Mathematics Education, Interdisciplinary Instruction, Pre-Service Teachers

Pre-service teachers (PSTs) often struggle with interdisciplinary lesson planning due to "limited content knowledge, accountability to meet content area standards, and limited self-efficacy in implementing integrated teaching" (Ryu et al., 2019, p.508). Ryu et al. (2019) also recommend PSTs use rubrics to analyze how materials may be used and use examples to demonstrate integration. Thus, the purpose of this article is to introduce the *Purposes of Mathematics Integration* framework that can be used to guide the next generation of mathematics teachers as they consider purposes of integrating mathematics lessons with other content area applications. This article also shares how using examples within the proposed framework helped PSTs align their interdisciplinary conceptualizations and lesson planning.

Interdisciplinary mathematics education is defined as the "conjunction of mathematics with other knowledge in problem solving and inquiry" (Williams & Roth, 2019, p. 14). The other knowledge in this context corresponds to one or more disciplines other than mathematics. For example, measurement and data can be connected with buoyancy in physics, water evaporation in life science, and plant growth in biology (An, 2017). There are several benefits of interdisciplinary instruction for learners and students: 1) it provides more effective learning opportunities for students (e.g., developing independent learning skills and in-depth conceptual understanding of multiple subjects; Berlin & White, 1999), 2) it helps meet the diverse needs of students and cultural responsiveness (Van Ingen et al., 2018), and 3) it allows teachers to teach subjects other than math and reading (Richards & Shea, 2006). More specifically, interdisciplinary STEM curricula in elementary grades contribute to positive changes in students' attitudes to learn multiple subjects via improvements in their engineering design skills (Chiang et al., 2020).

Theoretical Frameworks

We identified two trajectories to consider when evaluating interdisciplinary lesson planning in mathematics: the level of integration between mathematics and the other interdisciplinary content (IC) areas, as well as the organization of the integration. Huntley (1999) explains that there are five levels of integration (Figure 1).

Emphasis on Mathematics Increases

Math for Sake of Math	Math with IC	Math AND IC	IC with Math	IC for Sake of IC
Math lesson only	Interdisciplinary content establishes problem context or relevance (Connection)	Both content areas are explored at about the same level	Math establishes problem context or relevance of interdisciplinary content (Connection)	Interdisciplinary content lesson only
Comparison of metric units	Use actual distances of stars to compare metric units	Compare distance of stars using metric units and explore relationship between brightness of stars and distance	Remind students of metric units in math, then focus on relationship between brightness of stars and distances	Relationship between brightness of stars and distances

Emphasis on Interdisciplinary Content Increases

Figure 1. Level of Integration (Huntley, 1999).

Moving from left to right, the emphasis on mathematics within the interdisciplinary connection decreases and the emphasis on the interdisciplinary content increases. As the connections between mathematics and other disciplines are increasingly more related (i.e., moving towards the center), stronger interdisciplinary connections appear (Williams & Roth, 2019).

In addition to the level of integration, there are five different ways the math and interdisciplinary content can be organized within a lesson plan (see Table 1; Fogarty, 1991).

Table 1. Organization of Integration (Fogarty, 1991).

Type	Description	Examples
One subject only	Teaching Mathematics or IC only	Practicing addition/subtraction (in a mathematics lesson) OR only highlighting debit/credit entries (in a social studies lesson)
Threaded	Focusing on skills	Using technology (PowerPoint) or graphic organizer
Webbed	Using theme(s) to explore mathematics	Using the idea of social commerce to practice addition and subtraction
Sequence IC then Math	First teaching an IC then mathematics in a single lesson	First, sorting examples of debits and credits, and then discussing how we use addition and subtraction to help calculate debits and credits
Sequence Math then IC	First teaching mathematics then an IC in a single lesson	First practicing addition and subtraction, and then discussing how adding and subtracting help in social commerce
Shared	Implementing an activity where both content areas are needed to be successful	Purposefully selecting debits and credits, and using addition and subtraction to maintain balance for savings

When applying the five organizational strategies to mathematics, the lowest levels of integration focus on only one subject or specific skills that may cross multiple disciplines such as technology and graphic organizers. As the level of connection increases, the interdisciplinary content may be used as a theme to explore mathematical ideas. Sequencing the mathematics and interdisciplinary content areas focuses on one content area at a time, without clearly emphasizing the connections and relationships between the two areas. The highest level of organization of integration is a shared balance of both content areas in order to be successful with a single task.

We synthesized and merged Huntly's (1999) levels of integration and Fogarty's (1991) organization of integration in our newly conceptualized Purposes of Mathematics Integration framework in order to organize the relationships between the two trajectories into a single framework. This framework also adds a unique characterization for how the correlations between integration and organization bring forward four different purposes of mathematics integration for application or practice. The four purposes help build upon PSTs limited content knowledge and experiences to make connections between research and practical application (Figure 2).

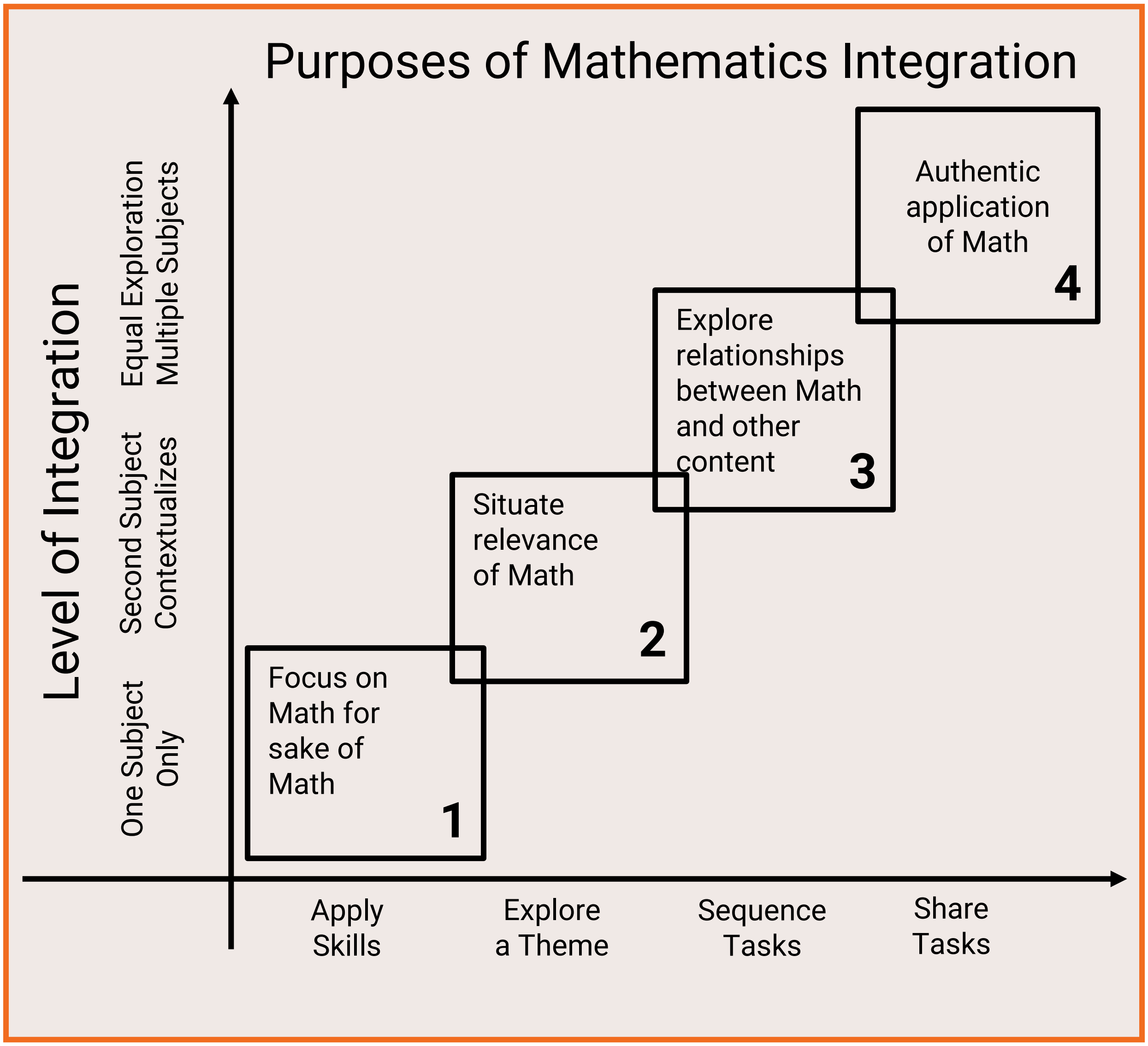

Figure 2. Purposes of Mathematics Integration framework.

In Figure 2, interdisciplinary connections increase along the y-axis by the level of integration (i.e., from one subject coverage to the use of a second subject for contextualization and finally to equal exploration of multiple subjects) and the x-axis by organization practice (i.e., application of skills, exploration of a theme, sequencing tasks, and sharing tasks).

As the quality of the interdisciplinary mathematics lessons increases along both trajectories, four purposes of the mathematics integration emerge: 1) focus on mathematics for the sake of mathematics, 2) situate relevance of mathematics, 3) explore relationships between mathematics and other content areas, and 4) explore authentic applications of mathematics.

We propose that our Purposes of Mathematics Integration framework can be used to evaluate how teachers conceptualize or use interdisciplinary connections in their lesson planning. It can also be used to help PSTs conceptualize in which conditions stronger or weaker interdisciplinary connections are most appropriate for their lesson planning. To illustrate how the framework can be used, we will share one context for data collection and evaluation.

Background and Procedures for Data Collection and Evaluation

Data were collected from four elementary mathematics methods courses across three universities and professors (Table 2). Course 3 and 4 had the same university and professor. All four courses had introductory content had an online discussion, and requested elementary PSTs to create an interdisciplinary lesson. However, there were differences in the introductory content for each course relating to the framework.

Table 2. Background of Courses and Participants

Course	Descriptions	Procedures
1 (n=11)	In-person graduate elementary mathematics methods course in a two-year program; some students are interns while others are teachers-of-record	• Introductory readings • Online Discussion • Interdisciplinary Lesson Plan
2 (n=12)	Hybrid in-person and online cross-listed elementary mathematics methods course for senior undergraduate and second semester graduate students in a one-year program.	• Introductory readings • Introductory arts integration video • Online Discussion • Interdisciplinary Lesson Plan
3 (n=12)	In-person senior undergraduate elementary mathematics methods course.	• Introductory readings • Introductory arts integration video • Online Discussion • Similarities/Differences 2 purposes • Interdisciplinary Lesson Plan
4 (n=12)	In-person senior undergraduate elementary mathematics methods course.	• Introductory readings • Introductory arts integration video • Online Discussion • Compare/Contrast Framework • Engage in Purpose 4 Activity • Interdisciplinary Lesson Plan

PSTs in courses 1-4 were given a short reading that situates interdisciplinary lesson planning within teaching through problem-solving (framework Purpose 1) and focuses on finding relevant contexts for mathematics (framework Purpose 2) (van de Walle et al., 2019). After the readings, PSTS in courses 2-4 were requested to view a video focusing on relationships between mathematics and the arts (framework Purpose 3).

Following the readings and videos, PSTs in all four courses engaged in an online discussion, which was to evaluate their conceptualizations of the purposes of integration based on the following prompt: What makes an elementary math lesson interdisciplinary? Discuss the benefits of interdisciplinary mathematics lessons and give some examples of ways to make a mathematics lesson plan interdisciplinary. They were also asked to respond to at least two other posts.

PSTs in course 3 engaged in an additional activity after the online discussion identifying similarities and differences between two purposes of integration. For example, PSTs compared and contrasted two lessons, one that dealt with the relevance of math (Purpose 2) and the other that explored relationships with a sequenced task (Purpose 3). In the lesson that addressed the relevance of math, students used different-sized land features to practice comparing whole numbers. In the lesson that explored relationships with a sequenced task, students identified different land features and their sizes on a map (social studies standard) and then compared and ordered them by size (math).

PSTs in Course 4, engaged in an additional activity after the online discussion conducting side-by-side comparisons between all four purposes of the framework. They also engaged in an activity requiring equal exploration of both mathematics and another grade-level content standard (Purpose 4). After engaging with all introductory content, PSTs in all four courses created their interdisciplinary lesson plans. The expectations for all course lesson plans were for PSTs to incorporate elements from at least two content areas to design an interdisciplinary mathematics lesson plan.

To categorize each PST's Purposes of Mathematics Integration as evidenced by their initial online discussion posts (completed after the reading and video), we first utilized the Level of Integration trajectory (Figure 1; Huntley, 1999) and the Organization of Integration trajectory (Table 1; Fogarty, 1991). Each PST's online discussion post was coded with (1) one type of Integration Level, and (2) one type of Integration Organization. Next, using the level and organization information, each PST's post was then coded with a Purpose of Integration (Figure 2). This same process was repeated to code PSTs lesson plans.

Evaluating Pre-Service Teachers' Interdisciplinary Math Conceptualizations and Planning

In this section, we discuss outcomes related to PSTs' interdisciplinary conceptualizations and lesson planning as evaluated using the Purposes of Mathematics Integration framework.

Pre-service Teachers' Conceptualizations of the Purposes of Integration

In Table 3, we provide the distribution of PSTs' Purposes of Integration conceptualizations from their online discussions, tabulated for each course/university as well as the total combined conceptualizations across all courses. Each cell in columns 2-5 represent the number of PSTs that exemplified a particular Purpose of Integration conceptualization for their respective course, with the final row representing the combined conceptualizations across all four courses.

Table 3. Pre-service Teachers' Conceptualizations of the Purposes of Integration.

	Purposes of Mathematics Integration (*n = 47*)				
Course	Purpose 1 Math only	Purpose 2 Relevance	Purpose 3 Relationship	Purpose 4 Application	All
Course 1	2	3	3	2	11
Course 2	3	3	3	3	12
Course 3	3	4	3	2	12
Course 4	3	5	4	1	12
Combined	11	15	13	8	47

As seen in Table 3, PSTs' initial conceptualizations after the video and short reading, and before any explicit introductions to the framework, are fairly similar across all four courses. The slight majority of PSTs (*n=15)* focused on using integration to situate the relevance of mathematics (Purpose 2). For example, one PST in Course 2 shared the following Purpose 2 conceptualization:

> *When we design an interdisciplinary math lesson, we are building a lesson that relates to one or more branches of knowledge... In order to make a math lesson interdisciplinary, teachers can use ideas from the topics that are being taught in other subjects that the students use including but not limited to language arts, social studies, or science. Furthermore, teachers can link topics of interest such as pop culture, sports, or arts...*

The next highest area (*n=13)* focused on the relationships and connections between content areas (Purpose 3). For example, one PST in Course 1 shared the following Purpose 3 conceptualization:

> *I think of interdisciplinary lessons as those that connect content from across multiple areas. It helps students think critically about how to connect what they are currently learning to what they have learned in the past in multiple areas. By connecting multiple content areas in one lesson, a good interdisciplinary lesson would help students understand what they are learning in school.*

Following close behind Purpose 3 was Purpose 1 (*n*=11), focusing on mathematics skills (Purpose 1). For example, one PST in Course 3 shared the following Purpose 1 conceptualization:

> *To define interdisciplinary lesson planning is to create a plan that draws different types of knowledge. It does not just refer to one branch. Relating this to math allows students to solve a problem in different ways. Math is an interdisciplinary subject.*

Only about one-sixth of PSTs (*n=8)* shared a Purpose 4 conceptualization focusing on authentic connections and applications. For example, one PST in Course 4 shared the following conceptualization.

> *Interdisciplinary to me means not just using two subjects together, but also making them connect. For example reading and social studies can be combined by reading a passage about a history topic. I feel like for this to be the most effective it is imperative that after the teacher or student reads the passage they also talk about the history that that book is discussing. For math this can be used a number of ways. The most common would be science or reading/writing. This is also only effective if you use both equally and not just reading a word problem and calling it a day. It is important to make sure when combining subjects such as math the other subject needs to also be showcased.[sic]*

These examples show how the Purposes of Interdisciplinary Lesson Planning can be used to understand PSTs' conceptualizations. These findings align with Ryu et al. (2019) that many PSTs have limited knowledge of interdisciplinary applications, with about a quarter of PSTs focusing on the lowest levels of integration and organization (Huntley, 1999; Fogarty, 1991). The examples align with research showing that when PSTs conceptualize integration, they often consider additional subjects beyond math and reading such as social studies, science, and the arts (Richards & Shea, 2006).

Evaluating Pre-service Teachers' Interdisciplinary Lesson Planning

Although the discussion showed similarities across PSTs' conceptualizations, the actual lesson plans designed by PSTs told a different story. The distribution of PSTs' Purposes of Integration within their interdisciplinary math lesson plans can be found in Table 4.

Table 4. Pre-service Teachers' Interdisciplinary Lesson Planning

Course	Purposes of Mathematics Integration (n = 47)				
	Purpose 1 Math Only	Purpose 2 Relevance	Purpose 3 Relationship	Purpose 4 Application	All
Course 1	11	0	0	0	11
Course 2	8	2	1	1	12
Course 3	4	2	3	1	12
Course 4	2	4	4	4	12
Combined	25	8	8	6	47

As seen in Table 4, Courses 1 and 2, which did not use examples to make comparisons between different purposes within the Purposes of Mathematics Integration framework, had primarily Purpose 1 lesson plans. For example, all 11 PSTs in Course 1 and 8 of 12 PSTs in Course 2 had lesson plans identified as focusing on mathematics skills only. The main activity description for one Purpose 1 lesson plan below focuses on the application of skills such as using context clues:

> *Students will be introduced to the multiplication 3rd graders with connections to real life and culture [sic]. Students are then reminded of basic multiplication facts, properties of multiplication, and Polya's 4 step process of problem solving. Students will be given a set of multiplication word problems and they will be asked to thoroughly "read" the problem. The teacher will display the word problems on the board and point out the "context words" to determine the multiplication sentence.*

Eight PSTs designed lessons that contextualized mathematics (Purpose 2). For example, the main activity description for one Purpose 2 lesson plan below uses the theme of economics to contextualize the purpose and relevance for money:

> *Students will skip count by 5s using nickels. Students will skip count by 10s using dimes. Students will skip count by 25s using quarters. We will then talk about how in our economy we use money to buy goods and services.*

Courses 3 and 4, which added activities to compare different purposes of lesson planning, showed increasingly stronger purposes of integration. As seen in Table 3, 4 of 12 PSTs in Course 3 and 8 of 12 PSTs in Course 4 focus on relationships (Purpose 3) or application (Purpose 4). For example, the main activity description for one Purpose 3 plan starts with a science lesson, links relationships between science and data from mathematics, before transitioning to mathematics to display/analyze data, and finally returning to the science topic:

> *As a class we will vote and predict what if they think each object is opaque, transparent, translucent. They will test their predictions and discuss their results. After getting the data the small group will make the chart/graph of their choice (Bar Graph, Pie chart) to represent the data. After they make their graphs, they can make predictions on what other objects, based on their finds, are opaque, transparent, and translucent.[sic]*

The main activity description for one Purpose 4 plan below has a single shared activity that combines the kindergarten physical education standard for throwing a ball and a mathematics standard for counting.

> *Students will be grouped into 5 groups of 4. Students will have 1-minutes [sic] to throw a ball underhand into a basket as many times as they can. Their rest of will use X marks on a piece of paper and count out load how many balls make it into a basket [sic]. One student will use a stopwatch and count down 1-minute with the guidance of the watch. Student counting down will change each round. The goal is for group to make it to 100 in the basket. If students exceed 100, they will push the basket further away to try a harder distance.*

In summary, when PSTs were asked to share their conceptualizations of interdisciplinary lesson planning, a general understanding of all four purposes was evident in the online discussions. However, without specific conversations comparing and contrasting the different purposes of integration, PSTs were more likely to focus on only the mathematics content or use the second content area to situate the relevance of mathematics within other disciplines or the real world when planning lessons. This supports Ryu et al.'s (2019) recommendation for using rubrics and examples to focus and extend PSTs' understanding of content and use of materials for interdisciplinary lesson planning. It also shows that using the framework can help move students beyond the math and reading focus shown in the Purpose 1 example toward other applications in science, social studies, and even physical education (Richards & Shea, 2006).

Implications for Teacher Education Programs

The outcomes from the above research show the potential for using the Purposes of Mathematics Integration framework to help the next generation of mathematics teachers to conceptualize and plan interdisciplinary mathematics lessons. This helps alleviate An's (2017) concerns that there are still gaps in the literature regarding effective methods of PST education to develop interdisciplinary mathematics instruction. Teacher education programs should consider when each purpose is appropriate for different types of lessons or assignments in their courses. This approach can better help PSTs understand the depth of conceptual understanding needed for the multiple subjects they teach (Berlin & White, 1999). Cavadas and his colleagues (2022) utilized their integration framework through problem-based learning activities. Building on their rationale, our Purposes of Mathematics Integration framework could be more effective if used with problem-based or model-eliciting activities by mathematics teacher educators. This framework can also help teacher educators evaluate how their PSTs conceptualize and integrate multiple content areas with mathematics; they can then use those evaluations to adapt their instruction and provide necessary content integration opportunities. Making specific comparisons between levels of integration in the Purposes of Mathematics Integration framework may help PSTs differentiate between the purposes and understand expectations.

Teacher education programs should also design and generate natural integration ideas in class discussions to consider authentic applications and connections or bring in experts from other content areas to support the generation of conceptualization and mathematical relationships with other content areas to expand PSTs limited P-12 content and standard knowledge (Richards & Shea, 2006). This may also help PSTs to teach more than just reading and math as well as make connections for deeper conceptual understanding of both integrated subjects (Berlin & White, 1999; Richards & Shea, 2006). Using the Purposes of Mathematics Integration framework can help PSTs see what they can do to move beyond just teaching math for the sake of learning math skills (Purpose 1) and start building relevant connections and conceptual understanding in small ways (Purpose 2), then sequence related tasks to explore relationships across subjects (Purpose 3), and finally engage in authentic applications of mathematics and other school curriculum (Purpose 4).

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